Twelf Tutorial

Twelf Encoding of Minilang

John Altidor

Motivation

- Proving language properties are important.
  - Rule out certain errors (e.g. assuming wrong number of bytes for an object).
  - Well-defined behavior throughout execution (e.g. no segmentation fault or accessing wrong parts of memory).
  - Publishing.

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  - Well-defined behavior throughout execution (e.g. no segmentation fault or accessing wrong parts of memory).
  - Publishing.
- But proofs are long, error prone, and difficult to validate.
  - **+20 pages** is common for a type safety proof.

Typical Proof Structure

- Example taken from type soundness proof of TameFJ calculus.

Lemma 33 (Inversion Lemma (method invocation)).

If:

- **a.** $\Delta;\Gamma \vdash \texttt{e.<}\overline{\texttt{P}}\texttt{>m(}\overline{\texttt{e}}\texttt{)} : \texttt{T} \mid \Delta'$
- **b.** $\emptyset \vdash \Delta$ OK
- **c.** $\Delta \vdash \Delta'$ OK
- **d.** $\forall \texttt{x} \in dom(\Gamma) : \ \Delta \vdash \Gamma(\texttt{x})$ OK

then:

there exists Δ_n

where:

$\Delta', \Delta_n = \Delta'', \overline{\Delta}$
$\Delta \vdash \Delta', \Delta_n$ OK
$\Delta;\Gamma \vdash \texttt{e} : \exists\Delta''\texttt{.N} \mid \emptyset$
$mType(\texttt{m}, \texttt{N}) = \texttt{<}\overline{\texttt{Y} \lhd \texttt{B}}\texttt{>}\overline{\texttt{U}} \to \texttt{U}$
$\Delta;\Gamma \vdash \overline{\texttt{e} : \exists\Delta\texttt{.R} \ \mid \emptyset}$
$match(sift(\overline{\texttt{R}}, \overline{\texttt{U}}, \overline{\texttt{Y}}), \overline{\texttt{P}}, \overline{\texttt{Y}}, \overline{\texttt{T}})$
$\Delta \vdash \overline{\texttt{P}}$ OK
$\Delta, \Delta'', \overline{\Delta} \vdash \overline{\texttt{T} <: [\overline{\texttt{T}/\texttt{Y}}]\texttt{B}}$
$\Delta, \Delta'', \overline{\Delta} \vdash \overline{\exists\emptyset\texttt{.R} <: [\overline{\texttt{T}/\texttt{Y}}]\texttt{U}}$
$\Delta, \Delta'', \Delta_n \vdash [\overline{\texttt{T}/\texttt{Y}}]\texttt{U} <: \texttt{T}$

Proof *by structural induction on the derivation of* $\Delta;\Gamma \vdash \texttt{e.<}\overline{\texttt{P}}\texttt{>m(}\overline{\texttt{e}}\texttt{)} : \texttt{T} \mid \Delta'$ *with a case analysis on the last step:*

Case 1 (T-Invk)

1.	$\Delta' = \Delta'', \overline{\Delta}$	*by def* T-Invk (1–2)
2.	$\texttt{T} = [\overline{\texttt{T}/\texttt{Y}}]\texttt{U}$	
3.	$\Delta;\Gamma \vdash \texttt{e} : \exists\Delta''\texttt{.N} \mid \emptyset$	*by premises* T-Invk (3–9)
4.	$mType(\texttt{m}, \texttt{N}) = \texttt{<}\overline{\texttt{Y} \lhd \texttt{B}}\texttt{>}\overline{\texttt{U}} \to \texttt{U}$	
5.	$\Delta;\Gamma \vdash \overline{\texttt{e} : \exists\Delta\texttt{.R} \ \mid \emptyset}$	
6.	$match(sift(\overline{\texttt{R}}, \overline{\texttt{U}}, \overline{\texttt{Y}}), \overline{\texttt{P}}, \overline{\texttt{Y}}, \overline{\texttt{T}})$	
7.	$\Delta \vdash \overline{\texttt{P}}$ OK	
8.	$\Delta, \Delta'', \overline{\Delta} \vdash \overline{\texttt{T} <: [\overline{\texttt{T}/\texttt{Y}}]\texttt{B}}$	
9.	$\Delta, \Delta'', \overline{\Delta} \vdash \overline{\exists\emptyset\texttt{.R} <: [\overline{\texttt{T}/\texttt{Y}}]\texttt{U}}$	
10.	*let* $\Delta_n = \emptyset$	
11.	$\Delta \vdash \exists\Delta''\texttt{.N}$ OK	*by* **3**, b, d, *lemma 30*
12.	$\Delta \vdash \Delta''$ OK	*by* **11**, *def* F-Exist
13.	$\Delta \vdash \overline{\exists\Delta\texttt{.R}}$ OK	*by* **5**, b, d, *lemma 30*
14.	$\Delta \vdash \overline{\Delta}$ OK	*by* **13**, *def* F-Exist
15.	$dom(\Delta'') \cap dom(\overline{\Delta}) = \emptyset$	*by* **3**, **5**, *Barendregt*
16.	$\Delta \vdash \Delta'', \overline{\Delta}$ OK	*by* **12**, **14**, **15**, *lemma 14*
17.	*done*	*by* **10**, **1**, **16**, **3**, **4**, **5**, **6**, **7**, **8**, **9**, **2**, *reflexivity*

- Lots of steps, lemmas, and opportunities for errors in proofs of language properties.

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  - These proofs must be specified in an extremely detailed, low-level form.
- **Proof assistants** are a hybrid of both.
  - "Hard steps" of proofs (the ones requiring deep insight) are provided by human.
  - "Easy steps" of proofs can be filled in automatically.
- (above bullet points taken from UPenn's Software Foundations course slides)

Twelf Proof Assistant

- Automated support for **deriving proofs** and **checking proofs** of language properties.
- Implementation of the LF calculus
  (calculus for reasoning about deductive systems).
- Alternatives: Coq, Isabelle, Agda, etc.
- Presenting Example Twelf Encoding of Minilang.

Constructive Logic

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- Proposition is true **iff** there exists a proof of it.
- Law of excluded middle not assumed: $P \vee \neg P$.
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- No choice operator ($\epsilon x.P(x)$) proposed by David Hilbert).
  - $ln(x) = u$ such that $x = e^u$.
  - Definition in Isabelle/HOL:

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definition ln :: real => real where
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- In Twelf: Writing a proof = Writing a program.
  - Proofs are programs.

Twelf Live Server

- Lecture will involve in-class exercises.
- Can try Twelf without installation.
- Twelf Live Server:

  http://twelf.org/live/

- Links to starter code of examples will be provided.

Kinds: Category of Types

- Three **levels** of objects in Twelf:
  - **Kinds** are at highest level.
  - **Types** are at second level.
  - **Terms** are at lowest level.

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- Contrived examples:
  - Term `[1, 2, 3]` is of type `ArrayInt`.
  - Type `ArrayInt` is of kind `Array`.

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  - Type `ArrayInt` is of kind `Array`.
- The kind `type` is a pre-defined kind in Twelf.

Functions

- Twelf supports defining functions:

```
int : type.   one : int.
plusOne : int -> int.
```

  - `plusOne` is a **function term**.
  - `plusOne` takes in a term of type `int` and returns a term of type `int`.
  - The type of function term `plusOne` is `int -> int`.
  - `(plusOne one)` has type `int`.

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- Functions taking in **multiple** arguments are represented using their **curried form**:

- `plus: int -> int -> int.`

  - `int -> int -> int` is curried form of `(int, int) -> int`.
  - `int -> int -> int` = `int -> (int -> int)`.
  - `(plus one)` has type `int -> int`.
  - `(plus one one)` has type `int`.

Functions Returning Types

- Recall that `type` is a **kind** (type of types).
- Functions can also return **types**:
- `equalsOne : int -> type.`
  - `equalsOne` is a function **term**.
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- `oneIsOne : (equalsOne one).`
  - Defines a new **term** `oneIsOne` of **type** `(equalsOne one)`.
- A function type is a kind if its return type is also a kind.
  - `int -> type` is a kind.
  - `int -> (int -> type)` is a kind.
  - `int -> int -> type` = `int -> (int -> type)` is a kind.
- `type` is **not** allowed on the left-hand side of arrow (->).

Minilang Syntax in Twelf

- The object language is Minilang (the object of study).
- Syntactic categories encoded w/ object types (defined types).
  - `exp : type.`
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  - `exp` represents syntactic category *e*.
  - Terms in the grammar of *e* have type `exp`.

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  - Terms in the grammar of e have type `exp`.
- Grammar productions encoded w/ **functions** between syntactic categories.
  - `add : exp -> exp -> exp.`
  - *Expression* $e ::= +(e_1; e_2)$
  - add takes in **two** arguments.
  - `exp -> exp -> exp` is curried form of `(exp, exp) -> exp`.

Terms w/ variables using Higher-Order Abstract Syntax (HOAS)

- Abstract syntax from earlier slides is **first-order abstract syntax (FOAS)**.
  - Each AST has form $o(t_1, t_2, \ldots, t_n)$, where o is operator and $t_1, \ldots, t_n$ are ASTs. Example:

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- ASTs in **Higher-Order Abstract Syntax (HOAS)**:
- Each t_i in $o(t_1, t_2, \ldots, t_n)$ has form:

$$x_1, x_2, \ldots x_k.t$$

- t is a FO-AST.
- Each x_j is a variable bound in t.
- $k \geq 0$; if $k = 0$, then no variable is declared.

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- HOAS lets us know where variables are being bound.

$$\texttt{let}(3;\ x.\texttt{+}(x;\ 4)) \equiv \texttt{let}(3;\ y.\texttt{+}(y;\ 4))$$

- Two preceding terms above are **alpha-equivalent**.

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- Holes **abstract** details.
- "$x.e$" represented by **lambda abstraction** "$\lambda x : \tau.e$".
- Twelf's syntax of "$\lambda x : \tau.e$": "`[`$x : \tau$`]` e"

`let` expression in Twelf HOAS

- Twelf type signature of `let`:

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- Example HOAS term in Twelf:

Concrete Syntax	*Twelf HOAS*
$\texttt{let x = } \underbrace{\texttt{1 + 2}}_{e_1} \texttt{ in } \underbrace{\texttt{x + 3}}_{e_2}$	$\texttt{let } \underbrace{\texttt{(add 1 2)}}_{e_1} \ \underbrace{\texttt{([x:exp] add x 3)}}_{x.e_2}$

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- No need to define object (Minilang) variables.
- **LF variables** remove need for object variables.
- No need to define substitution (nor requisite theorems) as well.

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  Functions that return types (not terms).
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  - `Vec(3)` is a **dependent type** representing 3-dimensional vectors.
  - `[4, 1, 3]` is a **term** of **type** `Vec(3)`.
  - `Vec` is a **type family** because it is a function that returns dependent types.

Judgments are Dependent Types

- Judgments/Propositions (instantiations of predicates) represented by dependent types.
- Judgment $z : \texttt{num}$ represented by type `(of (enat z) num)`.
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- Curry-Howard Correspondence:
  Proofs are terms.
  Propositions/Judgments are types.

Pi-Abstractions

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  `S -> T`
- Twelf Syntax for $\Pi x : S.T$:
  `{x:S} T`

Inference Rules are Functions

$$\frac{}{\texttt{num[}n\texttt{]} \;\texttt{:}\; \texttt{num}}\;\texttt{T.1}$$

- `of/nat : {N:nat} of (enat N) num.`
- Twelf Convention:
  - Constants start with lower-case letters.
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- Twelf Convention:
  - Constants start with lower-case letters.
  - Variables/parameters start with upper-case letters.
- `(of/nat z)` $\neq$ `(of (enat z) num)`.
- Function `of/nat` **returns terms** not types.
- Function `of` **returns types**.

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  - Constants start with lower-case letters.
  - Variables/parameters start with upper-case letters.
- `(of/nat z)` $\neq$ `(of (enat z) num)`.
- Function `of/nat` **returns terms** not types.
- Function `of` **returns types**.
- `(of/nat z)` = **term** of type `(of (enat z) num)`.
  - Example legal assignment:
    `y : (of (enat z) num) = (of/nat z).`

Inference Rules are Functions

$$\frac{}{\texttt{num}[n] \;:\; \texttt{num}}\;\texttt{T.1}$$

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- Function `of` **returns types**.
- `(of/nat z)` = **term** of type `(of (enat z) num)`.
  - Example legal assignment:
    `y : (of (enat z) num) = (of/nat z).`
- `(of/nat z)` is a **derivation/term** of judgment $z : \texttt{num}$ represented by type `(of (enat z) num)`.

Premises are Inputs

$$\frac{e_1 : \texttt{num} \quad e_2 : \texttt{num}}{\texttt{+}(e_1 ; \ e_2) : \ \texttt{num}}\ \texttt{T.4}$$

- Twelf Encoding:

```
of/add : of (add E1 E2) num
      <- of E1 num
      <- of E2 num.
```

- Given a proof of `(of E2 num)` **and**
- Given a proof of `(of E1 num)`
- `of/add` returns proof of `(of (add E1 E2) num)`

Implicit and explicit parameters

- `of/nat: {N:nat} of (enat N) num.`
- Parameter `N` is **explicit** in the above signature.
- Explicit parameters **must be specified** in function applications.
- `D : of (enat z) num = of/nat z.`

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- `of/nat: of (enat N) num.`
- Parameter `N` is **implicit** in the above signature.
- Implicit parameters **cannot be specified** by programmer in function applications.
- `D : of (enat z) num = of/nat.`
- Twelf figures out from the context (type of left-hand side of assignment) that `z` is the implicit parameter that `of/nat` should be applied to.

First-Order Quantification Only

- Can only quantify over first-order terms.
- **Allowed**:
  - `add :  exp -> exp -> exp.`
  - `let :  exp -> (exp -> exp) -> exp.`
    - A **higher-order term** is a function, where one of its inputs is also a function.

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- **Not allowed**:
  - `quantifyTypes : exp -> type -> exp.`
  - `allIsTrue : {Prop:type} Prop.`
- The kind `type` categorizes Twelf types.

First-Order Quantification Only

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- **Not allowed**:
  - `quantifyTypes : exp -> type -> exp.`
  - `allIsTrue : {Prop:type} Prop.`
- The kind `type` categorizes Twelf types.
- No type polymorphism implies no **general** logical connectives.
- **Not allowed**:

```
conjunction :
{P:type} {Q:type} P -> Q -> (and P Q).
```

Predicativity

- Different term levels used to restrict quantification.
  - Twelf terms are first-order terms; e.g., `(s z)`.
  - Twelf types are second-order terms; e.g., `nat`.
  - Twelf kinds are third-order terms; e.g., `type`.

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  - Twelf terms are first-order terms; e.g., `(s z)`.
  - Twelf types are second-order terms; e.g., `nat`.
  - Twelf kinds are third-order terms; e.g., `type`.
- Twelf only allows **predicative** definitions:
  - Cannot apply term to itself. (Cannot quantify over oneself.)
  - No term has itself as type. (Not allowed: `typ : typ`.)
  - Disallows Russell's paradox:
    Let $H = \{x \mid x \notin x\}$. Then $\underbrace{H \in H \iff H \notin H}_{\textit{False}}$.
- Helps Twelf avoid logical inconsistency
  (i.e. proving false/uninhabited type).
- False implies any proposition (including false ones).
- False/uninhabited types used for constructive proofs by contradiction.

Laboratory

- Create language of numbers with subtyping in Twelf.

Category	**Item**		**Abstract**	**Concrete**
Terms	e	$::=$	`zero`	0
		$\mid$	`pi`	π
		$\mid$	`img`	$\sqrt{-1}$
Types	t	$::=$	`number`	*num*
		$\mid$	`real`	*real*
		$\mid$	`complex`	*complex*
		$\mid$	`int`	*int*

More Exercises

- **Subtyping** Rules (not all):

$$\frac{}{complex <: num} \qquad \frac{}{real <: num} \qquad \frac{}{int <: real}$$

- **Typing** Rules (not all):

$$\frac{}{0 : int} \qquad \frac{}{\pi : real} \qquad \frac{}{\sqrt{-1} : complex}$$

- Define **reflexive** and **transitive** rules for subtyping.
- Define **subsumption** rule for typing judgment.
- **Prove** $0 : num$.
  - Fill in the blank below:
  - `D : (of zero number) =` •

Hypothetical Judgments in Twelf

- What happened to typing context Γ?

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  Function types where one of the inputs is also a function type.

Hypothetical Judgments in Twelf

- What happened to typing context Γ?
- **Hypothetical Judgments**:
  Judgments made under the assumption of other judgments.
- Encoded w/ **higher-order types**:
  Function types where one of the inputs is also a function type.
- Input function types represent hypothetical assumptions.
- Similar to higher-order terms.
  (Another application of HOAS)
- Γ does not need to be defined.

Typing `let` expression in Twelf HOAS

$$\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma, x : \tau_1 \vdash e_2 : \tau_2}{\Gamma \vdash \texttt{let}(x;\ e_1;\ e_2) :\ \tau_2}\ \texttt{T.6}$$

- Twelf Encoding:

```
of/let :  ({x:  exp} of x T1 -> of (E2 x) T2) ->
     of E1 T1 ->
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- Twelf Encoding:
```
of/let :  ({x:  exp} of x T1 -> of (E2 x) T2) ->
     of E1 T1 ->
     of (let E1 ([x] E2 x)) T2.
```
- First, a Twelf **coding convention**:
  Return type `(of (let E1 ([x] E2 x)) T2)` could be replaced with `(of (let E1 E2) T2)`.
- `E2` in both cases is of type `(exp -> exp)`.
- `([x] E2 x)` used for readability: # of inputs explicit.
- `([x] E2 x)` is called the **eta-expansion** of `E2`.

of/let's first input type

- Let f be a function of `of/let`'s first input type:
  `({x: exp} of x T1 -> of (E2 x) T2)`.
- That type models hypothetical judgment: $\Gamma, x : \tau_1 \vdash e_2 : \tau_2$.

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  dx = proof that f's first input has type T1

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- f's second input is a term `dx` of type `(of x T1)`.
  `dx` = proof that f's first input has type `T1`
- f's output is a term of type `(of (E2 x) T2)`:
  a proof that `E2` instantiated with `exp` term `x` has type `T2`.

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- f's output is a term of type `(of (E2 x) T2)`:
  a proof that `E2` instantiated with `exp` term `x` has type `T2`.
- `dx` of type `(of x T1)` can be **used in the definition of** f to return a proof/term of type `(of (E2 x) T2)`.
- The ability to use a proof (`dx`) of type `(of x T1)` to derive a proof of type `(of (E2 x) T2)` simulates the ability to use an assumption $x : \tau_1$ to prove $e_2 : \tau_2$.

Exercise Applying Hypothetical Judgment

- Derive the judgment $\vdash$ `let x be 1 in x + 0` : `num` in Twelf.
- Twelf encoding of above judgment:
  `of (let (enat (s z)) ([x:exp] add x (enat z))) num.`
- Recall important signatures (displaying implicit parameters):

```
of/let :  {T1:typ} {E2:exp -> exp} {T2:typ} {E1:exp}
  ({x:exp} of x T1 -> of (E2 x) T2) -> of E1 T1
    -> of (let E1 ([x:exp] E2 x)) T2.

of/nat : {N:nat} of (enat N) num.

of/add :  {E2:exp} {E1:exp}
  of E2 num -> of E1 num -> of (add E1 E2) num.
```

Solution to Previous Exercise

- Derive the judgment $\vdash$ `let x be 1 in x + 0` : `num` in Twelf.
- Twelf encoding of above judgment:
  `of (let (enat (s z)) ([x:exp] add x (enat z))) num.`
- Recall important signatures (without implicit parameters):

```
of/let : ({x:exp} of x T1 -> of (E2 x) T2) ->
  of E1 T1 -> of (let E1 ([x:exp] E2 x)) T2.
of/nat : of (enat N) num.
of/add :
  of E2 num -> of E1 num -> of (add E1 E2) num.
```

- Twelf proof of above judgment:

```
of/let
  ([x:exp] [dx:of x num] of/add (of/nat z) dx).
  (of/nat (s z))
```

Relations w/ Inputs and Outputs (Modes)

- Inputs/Outputs defined with `%mode` declaration.

  ```
  of :  exp -> typ -> type.
  %mode of +E -T.
  ```

- Inputs marked with `+`.
- Outputs marked with `-`.

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- Modes used also for **checking proofs of theorems**.

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- Not all relations required modes.
- Modes are necessary for **specifying theorems**.
- Modes used also for **checking proofs of theorems**.
- Only **ground** terms may be applied to relations w/ modes in rules (details later).

Backward Arrow vs. Forward Arrow

- Output terms must be **ground** given ground input terms.
  - Ground terms do not contain **free** variables.
  - Output terms are fixed (ground) wrt (ground) inputs.
- Forward "->" reflects **order that premises are passed to rules/functions** and makes proofs more natural.
- Backward "<-" reflects **order of resolving ground terms**.

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- Order of args that **causes error**:

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of/let : of E1 T1 ->
     ({x: exp} of x T1 -> of (E2 x) T2) ->
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```

- Error message:

```
Occurrence of variable T1 in output (-) argument
not necessarily ground
```

Universally-Quantified Inputs

- Terms in **input positions** of **return type** are **universally-quantified** inputs to function.
- `right1 :  of E1 num -> of (add E1 (enat z)) num.`

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- Terms in input position of return type: `(add E1 (enat z))`.
- Tokens starting with capital letters are assumed by Twelf to be variables in type: `E1`.
- Free variables in input position of return type, `E1`, are inferred by Twelf to be **universally**-quantified inputs to function `right1`.
  - Only these terms are allowed to be universal inputs to function `right1`.

Resolving Ground Terms

- All terms must be **ground** terms:
  constants or terms **without** free variables
  assuming that input terms (**from return type**) are also ground (do not contain free variables).
- Next Step:
  Check that input terms in type preceding return type are ground:
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- `E1` in premise type (`of E1 num`) is ground wrt `E1` in return type because they are the same.
- `num` in premise type (`of E1 num`) is ground wrt return type because `num` is a constant.

Non-ground Term in Premise Causing Error

- `wrong1 : of E2 num -> of (add E1 (enat z)) num.`
- E2 term not coming from conclusion (return type).

Ground Term From Output

- Output terms resulting from grounded input terms are also ground.
- Second argument of the `of` relation is an **output** argument.

```
right2 :  of E T -> of (add E (enat z)) T.
```

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- Term T is **computed**/result of premise/**recursive call** (`of E T`).

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```

- Term T is **computed**/result of premise/**recursive call** `(of E T)`.

Non-ground Term in Conclusion Causing Error

- Output term in conclusion not grounded:

```
wrong2 :  of E T1 -> of (add E (enat z)) T2.
```

- Output term T2 is universally quantified instead of a grounded result of the input term.
  This violates the `%mode` declaration of the `of` relation.

Previous Examples for Grounds Checking

- `right1 :  of E1 num -> of (add E1 (enat z)) num.`
- `wrong1 :  of E2 num -> of (add E1 (enat z)) num.`
- `right2 :  of E T -> of (add E (enat z)) T.`
- `wrong2 :  of E T1 -> of (add E (enat z)) T2.`

Decidable Predicate Definitions

- Decidable Predicate Definition or Algorithmic Definition: Definition of predicate that gives an **algorithm** for deciding predicate thats halts on all inputs within a **finite** number of steps.
- Constructive Logic Requirement: Proposition is true **iff** there exists a proof of it.

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- Constructive Logic Requirement: Proposition is true **iff** there exists a proof of it.
- For every true proposition/instance of predicate, algorithm finds a proof of proposition.
- For every false proposition of predicate, algorithm determines no proof exists.

Termination

- `%terminates` checks a program succeeds or fails in a finite number of steps given ground inputs.
- Modes with termination ensure decidable definitions.
- Termination not guaranteed with **transitive** rule.

```
subtype : typ -> typ -> type.
%mode subtype +T1 -T2.
subtype/int/rea : subtype int real.
subtype/rea/num : subtype real number.
subtype/num/num : subtype number number.
subtype/trans:
  subtype T1 T3 <- subtype T1 T2 <- subtype T2 T3.
%terminates T (subtype T _).
```

- **Error**: `Termination violation: ---> (T1) < (T1)`
- First input to subtype not **smaller** in premise/recursive call.

Syntax-Directed Definition

- **Syntax-Directed Definition**: For each syntactic form of input, there is at most one applicable rule.
- Syntax of input term tells us which rule to use. (or if no rule applies)
- Each true proposition of a syntax-directed predicate has exactly one unique derivation.
- Only one way to derive `+(5; 3) : num`.

$$\dfrac{\dfrac{}{\texttt{5 : num}}\,\texttt{of/num} \qquad \dfrac{}{\texttt{3 : num}}\,\texttt{of/num}}{\texttt{+(5; 3): num}}\,\texttt{of/add}$$

- No need for exhaustive proof search with syntax-directed predicates.

Checking Syntax-Directed

- Check that rules of relation/type family (e.g. `subtype`) are syntax-directed by passing relation to `%unique` declaration.
- `%unique` checks if output arguments of relation are uniquely determined by input arguments.
- `%unique` also checks if two rules overlap or can derive the same judgment.

```
subtype : typ -> typ -> type.
subtype/rea/num : subtype real number.
subtype/num/num : subtype number number.
subtype/trans:
  subtype T1 T3 <- subtype T1 T2 <- subtype T2 T3.
%worlds () (subtype _ _).
%unique subtype +T1 +T2.
```

- **Error**: `subtype/rea/num` and `subtype/trans` `overlap`
- Both rules could be used to derive `subtype real number`.

Automatic Proof Derivation

- Twelf can derive (search) for proofs:
- `%solve D1 :`
  ```
  of (estr (a , b , c , a , eps)) string.
  ```
- Twelf will save proof term in `D1`.

Printing Proof Terms

- To print all (implicit) terms in proofs:
- From Twelf Server:
  "`set Print.implicit true`"
- From ML (SML) Prompt:
  "`Twelf.Print.implicit := true`"
- Then just execute "`Check File`":
  Emacs Key Sequence: `^C ^S`

Proof Term in Sample Output

```
loadFile test_typing.elf
[Opening file test_typing.elf]
%solve
of (estr (, a (, b (, c (, a eps))))) string.
 OK
D1 : of (estr (, a (, b (, c (, a eps))))) string
   = of/str (, a (, b (, c (, a eps)))).
```

Twelf Theorems

- **Preservation Theorem**:
  If `(of E T)` and `(step E E')`, then `(of E' T)`.

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- **Preservation Theorem**:
  If `(of E T)` and `(step E E')`, then `(of E' T)`.
- Twelf allows expressing ∀∃-type properties.
- **Preservation, re-formulated**:
  - **For every** derivation of `(of E T)` and `(step E E')`,
  - **there exists** at least one derivation of `(of E' T)`.

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- **Preservation, re-formulated**:
  - **For every** derivation of `(of E T)` and `(step E E')`,
  - **there exists** at least one derivation of `(of E' T)`.
-
```
%theorem
preservation :
      forall* {E} {E'} {T}
      forall {O:of E T} {S:step E E'}
      exists {O':of E' T}
      true.
```
- Verbose syntax above.
  Desugared, concise alternative on next slide.

Theorems are Function Types w/ Specified Inputs/Outputs

- Preservation theorem is a function returning types (type family):

```
preservation:
     of E T -> step E E' -> of E' T -> type.
```

- Premises are **inputs**. Conclusions are **outputs**.

```
%mode preservation +O +S -O'.
```

- To prove preservation theorem, need to show `preservation` is a **total relation** on all **possible inputs**.
  - For each possible derivation of premises (inputs), need at least one derivation of conclusion (output).

Proofs of Theorems

- Proofs of theorems are **total** relations over inputs.
- Proving theorem
  = Constructing functions **for each case**:
  - For each **constructor** of term to perform structural induction on.

Proofs of Theorems

- Proofs of theorems are **total** relations over inputs.
- Proving theorem
  = Constructing functions **for each case**:
  - For each **constructor** of term to perform structural induction on.
- Note:
  No case-split or **pattern match** construct in Twelf.
  - This is the reason why **multiple functions** are required to prove theorem for multiple cases.
  - Results in smaller proof terms but more of them.

Preservation Proof - Addition Case 2 - Informal

Case: ($\mathtt{T.4}$, $\mathtt{D.1}$)

$$\frac{e_1 : \ \mathtt{num} \quad e_2 : \ \mathtt{num}}{\mathtt{+(}e_1\mathtt{;}\ e_2\mathtt{)} : \ \mathtt{num}}\ \mathtt{T.4} \qquad \frac{e_1 \mapsto e_1'}{\mathtt{+(}e_1\mathtt{;}\ e_2\mathtt{)} \mapsto \mathtt{+(}e_1'\mathtt{;}\ e_2\mathtt{)}}\ \mathtt{D.1}$$

We assume preservation holds for subexpressions. Hence, by the **inductive hypothesis**, $e_1 : \ \mathtt{num}$ and $e_1 \mapsto e_1'$ implies $e_1' : \ \mathtt{num}$. Rule $\mathtt{T.4}$ gives us:

$$\frac{e_1' : \ \mathtt{num} \quad e_2 : \ \mathtt{num}}{\mathtt{+(}e_1'\mathtt{;}\ e_2\mathtt{)} : \ \mathtt{num}}\ \mathtt{T.4} \qquad \square$$

Twelf Proof of Addition Case 2

```
of/add :
    of (add E1 E2) num <- of E1 num <- of E2 num.

- :
{E1-num  : of E1 num }
{E2-num  : of E2 num }
{E1=>E1' : step E1 E1' }
{E1'-num : of E1' num }
preservation E1-num E1=>E1' E1'-num ->
 preservation
  ((of/add E2-num E1-num) : (of (add E1 E2) num))
  ((step/add1 E1=>E1') :
      (step (add E1 E2) (add E1' E2)))
  ((of/add E2-num E1'-num) : (of (add E1' E2) num)).
```

Proof Case without Explicit Types

```
- : preservation
      (of/add E2-num E1-num)
      (step/add1 E1=>E1')
      (of/add E2-num E1'-num)
        <- preservation E1-num E1=>E1' E1'-num.
```

- Types of terms in proofs: usually not required to specify.
- Allowed to be manually specified.
- Output from Twelf server contains (some) inferred types.

Applying Inductive Hypothesis

```
- : preservation
      (of/add E2-num E1-num)
      (step/add1 E1=>E1')
      (of/add E2-num E1'-num)
        <- preservation E1-num E1=>E1' E1'-num.
```

- Applying inductive hypothesis = recursive call.

Checking Proof Totality

- After proving all cases, ask Twelf to check we covered all cases.

```
%worlds () (preservation _ _ _).
%total E-T (preservation E-T _ _).
```

- `%total E-T` tells Twelf to check proof of totality by structural induction on typing derivation `E-T`.
- Details of `%world` declaration later.

Missing Case

- If we forget to prove a case, `%total` command will fail.
- Twelf prints error message to help user "debug" proof:

```
preservation.elf:69.8-69.11 Error:
Coverage error --- missing cases:
{E1:exp} {E2:exp} {E3:exp}
{O1:of (add E1 E2) num} {S1:step E1 E3}
{O2:of (add E3 E2) num}
   |- preservation O1 (step/add1 S1) O2.
```

- Forgot the case where we could derive:
  - `(of (add E1 E2) num)`
  - `(step (add E1 E2) (add E3 E2)))`
- Need to construct proof of `(of (add E3 E2) num)`.

Assuming What Needs To Be Proven

- Cannot prove theorem by just assuming conclusion of theorem holds.

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Assuming What Needs To Be Proven

- Cannot prove theorem by just assuming conclusion of theorem holds.
- Also, cannot assume propositions not derived from premises of theorem.
- Such a proof will contain a **non-ground** term.
  - `%mode` declarations used to check proofs.

Recall Valid Proof of Case

```
- : {E1-num  : of E1 num}
    {E2-num  : of E2 num}
    {E1=>E1' : step E1 E1'}
    {E1'-num : of E1' num}
    preservation E1-num E1=>E1' E1'-num
      -> preservation (of/add E2-num E1-num)
                      (step/add1 E1=>E1')
                      (of/add E2-num E1'-num).
```

Invalid Proof of Case

```
- : {E1-num  : of E1 num }
    {E2-num  : of E2 num }
    {E1=>E1' : step E1 E1' }
    {E1'-num : of E1' num }
    preservation (of/add E2-num E1-num)
                 (step/add1 E1=>E1')
                 (of/add E2-num E1'-num).
```

- Proof above just assumes `of E1' num`, which is not one of the assumptions for the case.
- `E1'-num` is not an input term in the conclusion (third) argument of `preservation`.
- `E1'-num` is not an output term derived from ground terms.
- Twelf reports error for function above.

Checking Entire Proofs of Theorems

- Twelf checks proofs of theorems by verifying three key aspects:
  - Type checking – Proof of correct proposition
  - Grounds checking – Valid assumptions
  - Coverage checking – Proved all cases of theorem
- Next few slides describes Twelf's coverage checking of proofs

Specifying Worlds – Possible Inputs

- To check totality of function/theorem, need to define all possible inputs or **worlds**.
  - World = Set of terms of a type (inhabitants of a type)
- Example world of natural numbers:

```
nat : type.
z : nat.
s : nat -> nat.
%worlds () (nat).
```

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nat :  type.
z :  nat.
s :  nat -> nat.
%worlds () (nat).
```

- No term of type `nat` containing LF variables.
- No such `nat` of form `(s x)`, where `x` of variable of type `nat`.

Terms Containing Binders

- Let expression contains binders.

```
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let :  exp -> (exp -> exp) -> exp.
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syntax.elf:38.15-38.25 Error:
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```

- Need to tell Twelf about possible variables that can arise from rules.

Blocks

- **Blocks**: Patterns describing fragment of contexts.
- Update addressing previous error:

```
add :  exp -> exp -> exp.
let :  exp -> (exp -> exp) -> exp.
%block exp-block :  block {x:exp}.
%worlds (exp-block) (exp).
```

- Informs Twelf that terms of type `exp` can contain binders of type `exp`.

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%worlds (exp-block) (exp).
```

- Informs Twelf that terms of type `exp` can contain binders of type `exp`.
- Worlds can take in multiple blocks. Syntax:

```
%worlds (block1 | block2 | ... | blockN) (exp).
```

Worlds for Relations w/ Outputs

- Specifying world requires specifying how variables are quantified (universal inputs or ground outputs).

```
of : exp -> typ -> type.
%mode of +E -T.
...
of/let : of (let E1 ([x] E2 x)) T2
  <- of E1 T1
  <- ({x: exp} of x T1 -> of (E2 x) T2).
%block of-block :
  some {T:typ} block {x: exp}{_: of x T}.
%worlds (of-block) (of _ _).
```

- Number of args specified by pattern in `%worlds` declaration: `(of _ _)`

Checking Proof Totality

- After defining the worlds of all inputs to a theorem/function type, we can ask Twelf to check that the proof/function is total: **defined over the world**.

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- `%total E-T` tells Twelf to check proof of totality by structural induction on typing derivation `E-T`.
- Twelf checks proofs of theorems by:
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Totality Proof Automation

- Ask Twelf to **derive** proof for all cases of theorem:
  `%prove 3 E-T (preservation E-T _ _).`
  - by structural induction on typing derivation `E-T`
  - 3 is bound on the size of proof terms.

Totality Proof Automation

- Ask Twelf to **derive** proof for all cases of theorem:
  `%prove 3 E-T (preservation E-T _ _).`
  - by structural induction on typing derivation `E-T`
  - 3 is bound on the size of proof terms.
- Twelf fails to find proof of `progress` theorem because it requires nested case analysis.
  - Need extra theorems for sub-cases (no case-split construct).
  - See Twelf page on **Output Factoring** for more details: `http://twelf.org/wiki/Output_factoring`

My Review of Twelf: The Good

- **Language Simplicity**: Fewer language constructs
  - Functions encode many language elements (e.g., grammar, judgments, theorems, etc.)
- Good tool to **start with** for learning about proof assistants because of language simplicity and less syntactic sugar (my opinion).

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- **Language Simplicity**: Fewer language constructs
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- Good tool to **start with** for learning about proof assistants because of language simplicity and less syntactic sugar (my opinion).
- Language support (HOAS) for **variable binding**
  - Do not need to define substitution and prove substitution lemmas (sometimes).
- Language support for **context-sensitive propositions** (hypothetical judgments).
  - Do not need to define context of judgments and related lemmas (e.g. weakening) (sometimes).

My Review of Twelf: The Bad

- Language sometimes too simple
  - Missing support for frequent use-cases (e.g., nested case analysis, no case-split construct).
- Less verbosity can lead to cryptic code: Intent and meaning of code not clear without significant background:
  - No text suggesting this is a proof case:

    ```
    - : preservation (of/len _) (step/lenV _) of/nat.
    ```

  - No text suggesting this checks a proof of a theorem:

    ```
    %worlds () (preservation _ _ _).
    %total E-T (preservation E-T _ _).
    ```

- Error messages could be improved (e.g. missing cases messages).
  - Type annotations of function applications and defined names desired.

My Review of Twelf: The Bad (cont.)

- No support for stepping through proof instead of just reading proof trees.
- Lack of automation: Many proofs require manual specification (e.g. proofs requiring nested case analysis).
- No libraries.
  - No standard library
  - No import statements – All code must be included (repeat definition of `nat` for every project using them)
- No polymorphism
  - Separate definitions for `(int_list)`, `(str_list)`, etc.
  - Each type needs its own definition of equality.
- Many contexts require explicit definition (HOAS not always sufficient).

Summary

- Twelf is a proof assistant tool for checking and deriving proofs of properties of languages and deductive logics.
- A tool for language design and implementation.
- Imposes healthy reality and sanity check on language designs.
- Exposes, and helps correct, subtle design errors early in the process.